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Exponential Growth of Betti Numbers

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0. INTRODUCTION

Every ring in this paper is assumed to be commutative and noetherian.

Let $(R, \mathbf{n})$ be a local ring with maximal ideal $\mathbf{n}$ and M be a finitely generated R-module. The ith Betti number $b_i^R(M)$ of M is the integer $\dim_{R/n} \operatorname{Tor}_i^R(M, R/\mathbf{n})$, which is also the rank of the ith module in the minimal R-free resolution of M.

Our purpose in this paper is to study the sequence of Betti numbers over the local ring R, which is a homomorphic image of a local ring S. That is, $R \cong S/I$ for some ideal I of S. The expression $R \cong S/I$ is "natural" in the following sense. Let $\hat{R}$ be the $\mathbf{n}$-adic completion of R. Then $b_i^R(M) = b_i^{\hat{R}}(M \otimes_R \hat{R})$ for any finitely generated R-module M, since the $\mathbf{n}$-adic completion $\hat{R}$ of R is flat over R and $\mathbf{n}\hat{R}$ is maximal ideal of $\hat{R}$. So if a "good" property is true for the sequence of Betti numbers over $\hat{R}$, then it is also true over R. Accordingly, we may assume R is complete, then by Cohen structure theorem for complete local rings there exists a complete regular local ring $(S, \mathbf{m})$ such that $R \cong S/I$ for some ideal I of S.

Depending on I, the behavior of the sequence of Betti numbers has been known as follows:

THEOREM 0.1 [2, Theorem 4.1]. *If I is generated by a regular sequence. Then there exist real polynomials $p(x)$ and $q(x)$ of the same degree and with the same leading term such that*

$$p(i) \leqslant b_i^R(M) \leqslant q(i), \qquad i \gg 0.$$

For an ideal I of a ring S, let $\bar{I}$ of I denote the integral closure of I in S. That is, $\bar{I}$ is the set of all elements a in S satisfying equation of the form

$$a^m + \alpha_1 a^{m-1} + \cdots + \alpha_m = 0, \qquad \alpha_i \in I^i.$$

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DEFINITION 0.2. (1) Let $(S, \mathbf{m})$ be a local ring, I and ideal of S and let $J=(I :_S \mathbf{m})$. We say that I *satisfies* (H_0) if $\bar{I} \neq \bar{J}$, and that I *satisfies* (H_k) for $k \geqslant 1$ if for any prime p of height $\leqslant k$ and for any prime p with depth $S_p \leqslant 1$ $\overline{(I+p)/p} \neq \overline{(J+p)/p}$.

(2) A local ring $(R, \mathbf{n})$ is said to *satisfy* (H_k) for $k \geqslant 0$, if $R \cong S/I$ for some local ring S and an ideal I in S such that I satisfies (H_k).

Note that (H_k) implies (H_j) for $j=0, 1, ..., k$.

THEOREM 0.3 [4, Theorems 1.1 and 1.8]. *If I satisfies (H_k) for some $k \geqslant 0$, then*

$$b_{i+1}^R(M) - b_i^R(M) \geqslant k, \qquad i \geqslant 1.$$

Therefore over a complete intersection a question of Ramras (Problem 0.5) has a positive answer and Theorem 0.3 give examples of such an answer.

Problem 0.4 (L. Avramov; [1, (5.8)]). Is the sequence $b_i^R(M)$ eventually nondecreasing for any finitely generated module M over the local ring R?

Problem 0.5 (M. Ramras; [14]). It is true that for an arbitrary finitely generated module over a local ring R, there are only two possibilities: either the sequence $b_i^R(M)$ is eventually constant, or $\lim_i b_i^R(M) = \infty$?

Besides the possible asymptotic behavior of Betti numbers as stated above we study exponential growth of Betti numbers in this paper.

DEFINITION 0.6 [3, Definition 1.1]. Let M be a finitely generated module over the local ring $(R, \mathbf{n})$. We say that the sequence of Betti numbers has *exponential growth*, if there exist real numbers $1 < A \leqslant B$ such that we have the inequalities

$$A^i \leqslant \sum_{j=0}^{i} b_i^R(M) \leqslant B^i, \qquad i \gg 0.$$

If we can replace $\sum_{j=0}^{i} b_i^R(M)$ by $b_i^R(M)$ in the inequalities above, then we say the sequence of Betti numbers has *strong exponential growth*. An upper exponential bound B is known to exist for any finitely generated R-module (see, [10, Lemma in Sect. 1] or [3, (2.5)]).

In Section 1, we introduce an integer c associated with a ring R of the form S/I, such that for any finitely generated R-module M $cb_i^R(M) \leqslant b_{i+2}^R(M)$, $i \geqslant 1$ (Theorem 1.1). So if $c \geqslant 2$, then for any finitely generated non-free R-module M the sequence $b_i^R(M)$ has strong exponential growth. For example, if edim $S \geqslant 2$ and I is $\mathbf{m}$-primary $\mathbf{m}$-full ideal of S contained in $\mathbf{m}^2$, then $c \geqslant 2$. This c can be maximized by using a regular presentation, $S \to R \to 0$. In this case, c has a binomial lower bound (Theorem 3.5). Note that $\mathbf{m}$-full ideals form a larger class than integrally closed ideals due to a theorem of Goto (Theorem 3.4).

In Section 2, we prove that c is invariant under any regular presentation S of R (Theorem 2.4(2)).

1. Exponential Growth of Betti Numbers

In this paper [10], Lescot studied rings of the form $S/\mathbf{m}J$ as examples of Problem 0.5 and those of exponential growth of Betti numbers. If we consider rings of the form S/I and exploit the properties of the syzygy module K and its lifting L, then an integer c is derived for strong exponential growth of Betti numbers.

Theorem 1.1. *Let $(S, \mathbf{m})$ be a local ring and $R = S/I$ for an ideal I of S. Put $J = (I :_S \mathbf{m})$ and $c = \dim_{S/\mathbf{m}} \mathbf{m}J/\mathbf{m}I$. Then for any finitely generated R-module M*

$$cb_i^R(M) \leqslant b_{i+2}^R(M), \qquad i \geqslant 1.$$

Proof. We may assume that $c \neq 0$ and M is not free, otherwise the assertion is trivial. Let $\tilde{R} = S/\mathbf{m}I$, $b = b_i^R(M)$ and $K = \operatorname{syz}_R^{i+1}(M)$. Let $\pi\colon \tilde{R}^b \to R^b$ be the natural projection and $L = \pi^{-1}(K)$. Note that $K \subseteq \mathbf{m}R^b$ implies $L \subseteq \mathbf{m}\tilde{R}^b$, hence $IL = 0$, and L is an R-module. Note that $K \cong L/I\tilde{R}^b$ and $K/\mathbf{m}K \cong L/(I\tilde{R}^b + \mathbf{m}L)$. Thus

$$\begin{aligned}\mu(L) &= \dim(L/\mathbf{m}L)\\ &= \mu(K) + \dim(I\tilde{R}^b + \mathbf{m}L)/\mathbf{m}L.\end{aligned}$$

Consider the short exact sequence of R-modules,

$$0 \to (I/\mathbf{m}I)^b \to L \to K \to 0.$$

This gives a long exact sequence on $\operatorname{Tor}^R(—, k)$ (k is the residue field $S/\mathbf{m}$)

$$\operatorname{Tor}_1^R(K, k) \to \operatorname{Tor}_0^R((I/\mathbf{m}I)^b, k) \to \operatorname{Tor}_0^R(L, k) \to \operatorname{Tor}_0^R(K, k) \to 0. \quad (1)$$

Applying the length function l for (1) we obtain

$$\begin{aligned}b_{i+2}^R(M) &= l(\operatorname{Tor}_1^R(K, k))\\ &\geqslant b \dim(I/\mathbf{m}I) + \mu(K) - \mu(L)\\ &= b \dim(I/\mathbf{m}I) - \dim(I\tilde{R}^b + \mathbf{m}L)/\mathbf{m}L.\end{aligned}$$

Note now that in the minimal free resolution of M over R one has

$$d_i(JR^b) = Jd_i(R^b) \subseteq J\mathbf{m}R^{b_{i-1}} \subseteq IR^{b_{i-1}} = 0,$$

Hence $J\tilde{R}^b \subseteq L$. This implies that $(I\tilde{R}^b + \mathbf{m}L)/\mathbf{m}L \cong I\tilde{R}^b/(I\tilde{R}^b \cap \mathbf{m}L)$ is a homomorphic image of $I\tilde{R}^b/(I\tilde{R}^b \cap \mathbf{m}J\tilde{R}^b) = I\tilde{R}^b/\mathbf{m}J\tilde{R}^b$, hence $\dim(I\tilde{R}^b + \mathbf{m}L)/\mathbf{m}L \leqslant b \dim(I/\mathbf{m}J)$. Therefore

$$b^R_{i+2}(M) \geqslant b \dim(I/\mathbf{m}I) - b \dim(I/\mathbf{m}J) = cb^R_i(M). \quad \blacksquare$$

From now on c will denote $\dim \mathbf{m}J/\mathbf{m}I$ under the hypothesis of Theorem 1.1.

Remark 1.2. (1) If $c = 1$, then for any finitely generated R-module M the sequences $b^R_{2i}(M)_{i \geqslant 1}$ and $b^R_{2i+1}(M)_{i \geqslant 0}$ are nondecreasing. In this case Problem 0.5 has a positive answer, that is, the sequence $b^R_i(M)$ is eventually constant, or $\lim_i b^R_i(M) = \infty$ (see [10, Lemma on p. 292]).

(2) If $c \geqslant 2$, then for any finitely generated non-free R-module M the sequence $b^R_i(M)$ has strong exponential growth with a lower exponential bound A for any $1 < A < \sqrt{c}$.

PROPOSITION 1.3. *Let $(S, \mathbf{m})$ be a local ring and $R = S/\mathbf{m}J$ for an ideal J of S.*

(1) *If $\mathbf{m}J \neq 0$, then $c \geqslant 1$.*

(2) *If J is $\mathbf{m}$-primary, then $c \geqslant \dim S$.*

Proof. Let $I = \mathbf{m}J$ and $J_1 = (I : \mathbf{m})$. Then

$$c = \dim(\mathbf{m}J_1/\mathbf{m}I) \geqslant \dim(\mathbf{m}J/\mathbf{m}^2J). \tag{2}$$

(1) Suppose $c = 0$, then $\mathbf{m}J = \mathbf{m}^2J$ by (2). By Nakayama's lemma $\mathbf{m}J = 0$, which contradicts the hypothesis of (1).

(2) Suppose $c \leqslant \dim S - 1$. Then the hypothesis of (2) implies that

$$\mathbf{m}J = \mathbf{m}^2J + (f_1, ..., f_c), \qquad f_i \in \mathbf{m}.$$

This forces $\mathbf{m}J \subseteq (f_1, ..., f_c)$. This is a contradiction since J is $\mathbf{m}$-primary and $\mathrm{ht}(f_1, ..., f_c) \leqslant \dim S - 1$. $\blacksquare$

Due to Remark 1.2, a theorem of Lescot [10, Theorem A.3] is reproved (Proposition 1.3(1)) and examples of strong exponential growth have been provided (Proposition 1.3(2)).

The next result complements Theorem 0.3.

PROPOSITION 1.4. *Let $(S, \mathbf{m})$ be a local ring $R = S/I$ for an ideal I of S. If I satisfies (H_k) for some $k \geqslant 1$, then $c \geqslant k + 1$. In particular, for any* *finitely generated non-free R-module M the sequence $b_i^R(M)$ is strictly increasing and has strong exponential growth with a lower exponential bound A for any* $1 < A < \sqrt{k+1}$.

Proof. Suppose $c = \dim \mathbf{m}J/\mathbf{m}I \leqslant k$. Then we can express

$$\mathbf{m}J = \mathbf{m}I + (f_1, ..., f_c).$$

Let p be a prime containing $(f_1, ..., f_c)$ and of height $\leqslant k$. Then $\mathbf{m}J + p = \mathbf{m}I + p$. Hence

$$\overline{(J+p)/p} = \overline{(I+p)/p}$$

(use the determinant trick together with the fact that maximal ideal $\mathbf{m}/p$ is a faithful S/p-module). This contradicts (H_k). ∎

Remark 1.5. If I is an irreducible $\mathbf{m}$-primary ideal of a local ring $(S, \mathbf{m})$ (that is, $R = S/I$ is a 0-dimensional Gorenstein ring) and $\mu(\mathbf{m}/I) \geqslant 2$, then $c = 0$. In other words, $\mathbf{m}J = \mathbf{m}I$ where $J = (I : \mathbf{m})$. This is due to an example of Eisenbud [5, Sect. 3]: If $(R, \mathbf{n})$ is a 0-dimensional Gorenstein local ring with embedding dimension at least 2, then the Betti numbers $b_i^R(M_k)$ of $M_k = \mathrm{Hom}_R(\mathrm{syz}^k(R/\mathbf{n}), R)$ are strictly decreasing for $i = 0, ..., k-1$. So if we let $k \geqslant 4$, then

$$b_1(M_k) > b_2(M_k) > b_3(M_k),$$

which is impossible if $c \geqslant 1$.

The following examples show the independence of the two properties, $c = 1$ and (H_0).

EXAMPLE 1.6. (1) Let $S = k[[x, y]]$, the power series ring in two variables x and y over the field k, and let $I = (x^5, x^4y^5, y^6)$. Then $J = (x^5, x^4y^4, x^3y^5, y^6)$, $\mathbf{m}I = (x^6, x^5y, xy^6, y^7)$ and $\mathbf{m}J = (x^6, x^5y, x^4y^5, xy^6, y^7)$. Hence $\dim \mathbf{m}/\mathbf{m}I = 1$. Whereas $\bar{I} = \bar{J}$, since x^4y^4 and x^3y^5 are integral over I.

(2) Let $S = k[[x, y]]$ and let $I = (x^4, x^3y^2, y^5)$. Then $J = (x^4, x^3y, x^2y^4, y^5)$, $\mathbf{m}I = (x^5, x^4y, x^3y^3, xy^5, y^6)$ and $\mathbf{m}J = (x^5, x^4y, x^3y^2, xy^5, y^6)$. Hence $\dim \mathbf{m}J/\mathbf{m}I = 1$. Since x^3y is not integral over I, $\bar{I} \neq \bar{J}$.

2. THE INVARIANCE OF TEST MODULES

When $(S, \mathbf{m})$ and $(R, \mathbf{n})$ are local rings with an epimorphism $f: S \to R$, $I = \ker f$ and $J = (I :_S \mathbf{m})$, we write $C_R(S, f)$ for $\bar{J}/\bar{I}$ and $c_R(S, f)$ for $\dim \mathbf{m}J/\mathbf{m}I$. For obvious reason (Theorem 0.3 and Theorem 1.1), call $\bar{J}/\bar{I}$ a *test module for nondecreasing of Betti numbers and* $\mathbf{m}J/\mathbf{m}I$ *a test module for strong exponential growth of Betti numbers.*

In this section, we study the invariance of $C_R(S\cdot f)$ and $c_R(S\cdot f)$.

In [4, Theorem 2.5], we partially established the invariance of the module $\bar{J}/\bar{I}$. L. Avramov has suggested to use "fiber product" to connect any two regular presentation of the local ring R. He then gives a complete answer for the invariance of the module $\bar{J}/\bar{I}$. The same idea also helps to obtain a complete answer for the invariance of $\dim \mathbf{m}J/\mathbf{m}I$ under any regular presentation S of R.

I express my gratitude to L. Avramov for his ideas and help.

LEMMA 2.1. *Let* $(S_i, \mathbf{m}_i)_{i=1,2}$ *and* $(R, \mathbf{n})$ *be local rings with epimorphisms* f *and* g

$$S_2 \xrightarrow{f} S_1 \xrightarrow{g} R.$$

(1) *There is a surjection* $C_R(S_2, g\circ f) \to C_R(S_1, g)$, *which becomes an isomorphism when both* S_1 *and* S_2 *are regular.*

(2) $c_R(S_2, g\circ f) \geqslant c_R(S_1, g)$, *and the equality holds if both* S_1 *and* S_2 *are regular.*

Proof. (1) [4, Lemmas 2.1 and 2.3].

(2) Let $I_1 = \ker g$, $I_2 = \ker(g\circ f)$ and $J_i = (I_i :_{S_i} \mathbf{m}_i)_{i=1,2}$. Note that $f(\mathbf{m}_2 I_2) = f(\mathbf{m}_2)\, f(I_2) = \mathbf{m}_1 I_1$ and $f(\mathbf{m}_2 J_2) = f(\mathbf{m}_2)\, f(J_2) = \mathbf{m}_1 J_1$. Hence f induces a surjective homomorphism of vector spaces

$$\bar{f}\colon \mathbf{m}_2 J_2/\mathbf{m}_2 I_2 \to \mathbf{m}_1 J_1/\mathbf{m}_1 I_1.$$

If both S_1 and S_2 are regular, then it is enough to prove the lemma when $\dim S_2 = \dim S_1 + 1$. Then $\ker f = (x)$ for $x \in \mathbf{m} - \mathbf{m}^2$. Obviously, $x \in I_2$. Let $\mathbf{m}_2 = (x_1, ..., x_d)$ where $d = \dim S_2$. We may assume that $x = x_1$. Note that

$$\ker \bar{f} = \mathbf{m}_2 I_2 + (\mathbf{m}_2 J_2 \cap (x_1)).$$

However,

$$(x_1) \cap \mathbf{m}_2 J_2 \subseteq (x_1) \cap (x_1, ..., x_d)^2 = (x_1^2, x_1 x_2, ..., x_1 x_d) \subseteq \mathbf{m}_2 I_2.$$

Therefore $\ker \bar{f} = \mathbf{m}_2 I_2$ and the desired conclusion of the lemma follows from this. ∎

We recall the fiber product of rings [8, 0,18.1.2], which will connect any two regular presentations of R.

DEFINITION 2.2. Let S_i and R be rings (not necessarily neotherian) and $f_i: S_i \to R$ be homomorphisms of rings ($i = 1, 2$). Then the *fiber product* of S_1 and S_2 over R through f_1 and f_2, denoted by $S_1 \times_R S_2$, is the subring of the ring of the Cartesian product $S_1 \times S_2$ such that

$$S_1 \times_R S_2 = \{(x, y) \mid f_1(x) = f_2(y),\ x \in S_1,\ y \in S_2\}.$$

The restrictions $p_i: S_1 \times_R S_2 \to S_i$ of the projections $\pi_i: S_1 \times S_2 \to S_i$ are called the *canonical projections* of the fiber product $S_1 \times_R S_2$ ($i = 1, 2$).

The fiber product can be characterized by the following property: For a ring C and ring-homomorphisms $q_i: C \to S_i$ such that $q_1 \circ f_1 = q_2 \circ f_2$, there exists a unique homomorphism $g: C \to S_1 \times_R S_2$ such that $q_i = p_i \circ g$ ($i = 1, 2$).

LEMMA 2.3 [8, 19.3.2.1]. *Let S_i and R be rings and $f_i: S_i \to R$ be homomorphisms of rings ($i = 1, 2$).*

(1) *If f_1 is surjective and S_1 and S_2 are noetherian, then $S_1 \times_R S_2$ is noetherian.*

(2) *If S_i and R are local rings and f_i are local homomorphisms, then the canonical projections $p_i: S_1 \times_R S_2 \to S_i$ are local ($i = 1, 2$).*

(3) *If the conditions of* (1) *and* (2) *are satisfied and f_2 is surjective, and if S_1 and S_2 are complete noetherian local rings, then $S_1 \times_R S_2$ is a complete noetherian local ring.*

THEOREM 2.4. *Let $(S_i, \mathbf{m}_i)$ be complete regular local rings and $(R, \mathbf{n})$ be a complete local ring with epimorphisms $f_i: S_i \to R$ ($i = 1, 2$). Then*

(1) $C_R(S_1, f_1) \cong C_R(S_2, f_2)$.

(2) $c_R(S_1, f_1) = c_R(S_2, f_2)$.

Proof. By Lemma 3.6, the fiber product $S_1 \times_R S_2$ is a complete local ring with the surjective canonical projections $p_i: S_1 \times_R S_2 \to S_i$. Applying the Cohen structure theorem for complete local rings, we have a regular local ring S which maps onto $S_1 \times_R S_2$ by g. Hence we have the following diagram

$$S \xrightarrow{g} S_1 \times_R S_2 \xrightarrow{p_i} S_i \xrightarrow{f_i} R, \qquad i = 1, 2.$$

Note that f_i, p_i and g are all surjective. Now Lemma 2.1 implies that

$$C_R(S_1, f_1) \cong C_R(S, f_1 \circ p_1 \circ g) = C_R(S, f_2 \circ p_2 \circ g) \cong C_R(S_2, f_2),$$

$$c_R(S_1, f_1) = c_R(S, f_1 \circ p_1 \circ g) = c_R(S, f_2 \circ p_2 \circ g) = c_R(S_2, f_2).$$

This finishes the proof of the theorem. ∎

In view of the preceding result, we can now make the following definition:

DEFINITION 2.5. If R is a local ring, and $f: S \to \hat{R}$ is a Cohen presentation with S regular, then we set $C_R = C_{\hat{R}}(S, f)$ and $c_R = c_{\hat{R}}(S, f)$.

THEOREM 2.6. *Let $f: S \to R$ be an epimorphism of local rings.*

(1) *If $C_R(S, f) \neq 0$, then $C_R \neq 0$.*

(2) $c_R(S, f) \leqslant c_R$.

Proof. Since by definition $C_R \cong C_{\hat{R}}$ and $c_R = c_{\hat{R}}$, we can assume both R and S are complete. Let $T \to S$ be a Cohen presentation with T regular local. Both statements now follow from Lemma 2.1. ∎

3. m-FULL IDEALS

In this section we put more effort to broaden examples of strong exponential growth of Betti numbers. For the purpose of this we introduce some ideas initiated by D. Rees and studied by J. Watanabe and S. Goto.

DEFINITION 3.1. Let I be an ideal of a local ring $(S, \mathbf{m})$. Then

(1) Assume first that the residue field $S/\mathbf{m}$ is infinite. Then I is called $\mathbf{m}$-*full*, if $I = (\mathbf{m}I : x)$ for some $x \in S$. When $S/\mathbf{m}$ is not necessarily infinite, let S' be a local ring which is faithfully flat over S with infinite residue field and with $\mathbf{m}S'$ as maximal ideal. Then I is called $\mathbf{m}$-*full* if IS' is $\mathbf{m}S'$-full.

(2) I is said to have *the Rees property*, if $\mu_S(I) \geqslant \mu_S(J)$ for any ideal J containing I.

LEMMA 3.2. *If $(\mathbf{m}I : x) = I$, then $(I : \mathbf{m}) = (I : x)$.*

Proof. Obviously $(I : \mathbf{m}) \subseteq (I : x)$, while

$$(I : \mathbf{m}) = (\mathbf{m}I : x) : \mathbf{m} = (\mathbf{m}I : \mathbf{m}) : x \supseteq (I : x). \quad ∎$$

THEOREM 3.3 [15, Theorem 3]. *Let $(S, \mathbf{m})$ be a local ring. Then an $\mathbf{m}$-primary $\mathbf{m}$-full ideal has the Rees property.*

THEOREM 3.4 [7, Theorem 2.4]. *Let $(S, \mathbf{m})$ be a local ring with $S/\mathbf{m}$ infinite. If an ideal I is integrally closed, then I is $\mathbf{m}$-full or $I = \sqrt{(0)}$.*

In view of the preceding result, the following theorem complements Theorem 0.3.

For a local ring $(R, \mathbf{n})$ recall that edim $R = \dim_k \mathbf{n}/\mathbf{n}^2$, and set

$$o(R) = \sup\left\{i \mid l(R/\mathbf{n}^i) = \binom{\operatorname{edim} R + i - 1}{\operatorname{edim} R}\right\}.$$

Note that $o(R) \geqslant 2$, and that $o(R) = \infty$ if and only if R is regular.

THEOREM 3.5. *Let $(S, \mathbf{m})$ be a local ring of embedding dimension $d \geqslant 2$ and $R = S/I$ for some $\mathbf{m}$-primary $\mathbf{m}$-full ideal $I \subseteq \mathbf{m}^2$. Then $c_R \geqslant \binom{\operatorname{edim} R + o(R) - 2}{\operatorname{edim} R - 1}$. In particular, for any finitely generated non-free R-module M the sequence $b_i^R(M)$ has strong exponential growth with lower exponential bound A, for any A such that $1 < A = \sqrt{\binom{\operatorname{edim} R + o(R) - 2}{\operatorname{edim} R - 1}}$.*

Proof. Suppose $S/\mathbf{m}$ is finite, and let $(S', \mathbf{m}S')$ be a faithfully flat extension of S with infinite residue field and IS' is $\mathbf{m}S'$-full. Put $R' = S'/IS'$. Then R' is isomorphic to $R \otimes_S S'$ and is faithfully flat over R. Thus

$$b_i^R(M) = b_i^{R'}(M \otimes_R R').$$

So we may assume $S/\mathbf{m}$ is infinite and $I = \mathbf{m}I : x$ for some $x \in S$.

It is clear from Definition 3.1 that the ideal $I\hat{S}$ of the completion of S is $\mathbf{m}\hat{S}$-full and it is clearly $\mathbf{m}\hat{S}$-primary. Since $\hat{R} = \hat{S}/I\hat{S}$, we can furthermore assume R and S are complete. Let $S' \to S$ be a Cohen presentation of S with $(S', \mathbf{m}')$ a regular local ring and $\ker(S' \to S) = I''$, let I' be the inverse image of I in S', and let x' be an element of $\mathbf{m}'$ which maps to x. By assumption one has $I'/I'' = (\mathbf{m}(I'/I'') : x) = ((\mathbf{m}'I' + I''/I'') : x) = ((\mathbf{m}'I' + I'') : x')/I''$, hence $I' = ((\mathbf{m}'I' + I'') : x') \supseteq (\mathbf{m}'I' : x') \supseteq I'$. This shows that I' is $\mathbf{m}'$-full, and since it is also $\mathbf{m}'$-primary we can assume S is regular.

From the short exact sequences

$$0 \to \mathbf{m}J/\mathbf{m}I \to I/\mathbf{m}I \to I/\mathbf{m}J \to 0,$$
$$0 \to I/\mathbf{m}J \to J/\mathbf{m}J \to J/I \to 0,$$

we deduce

$$l(\mathbf{m}J/\mathbf{m}I) = \mu(I) - \mu(J) + l(J/I).$$

By Theorem 3.3 $\mu(I) \geqslant \mu(J)$, and by Lemma 3.2 $J = (I : \mathbf{m}) = (I : x)$ for some $x \in S$. Therefore

$$c_R = l(\mathbf{m}J/\mathbf{m}I) \geqslant l(I : x/I) = l(S/(I, x)).$$

Note that $x \in \mathbf{m} - \mathbf{m}^2$. Otherwise,

$$I = \mathbf{m}I : x \supseteq \mathbf{m}I : \mathbf{m}^2 \supseteq I : \mathbf{m} \supseteq I,$$

which is impossible since I is an $\mathbf{m}$-primary ideal. Thus $S/(x)$ is a regular local ring of dimension $\operatorname{edim} R-1$. Since $I\subseteq \mathbf{m}^{o(R)}$, $l(S/(I,x))\geqslant l(S/(\mathbf{m}^{o(R)},x))=\binom{\operatorname{edim} R+o(R)-2}{\operatorname{edim} R-1}$.

The last assertion of the theorem now follows from Theorem 1.1. ∎

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